III

ARITHMETICAL PROPERTIES OF THE SEQUENCE OF EXPONENTS

The properties of the function $f(s)$ represented by the series (1) depend on both sequences $\{\lambda_n\}$ and $\{a_n\}$. The properties of $\{\lambda_n\}$ which are mostly involved in the modern theory of Dirichlet series, are of two different kinds: a) the arithmetical character of the sequence; b) the growth of the number $N(t)$ of quantities λ_n smaller than t as t tends to infinity.

We shall begin with some theorems on Dirichlet series in which the λ_n have simple arithmetical properties.

Let us begin by studying a certain analytical transformation.

In the z-plane we define the region $\Delta(\alpha)$, $(\alpha>0)$ as follows: the frontier $C(\alpha)$ of $\Delta(\alpha)$ is composed of the segments

$$\left.\begin{array}{c} y=(2k+1)\pi \\ -\alpha\leqq x\leqq 0 \end{array}\right\} (k=0,\ \pm 1,\ \pm 2,\ \cdots;\ z=x+yi), \tag{18}$$

and the curve

$$x=-\alpha\sin^2\left(\frac{y}{2}\right), \tag{19}$$

$\Delta(\alpha)$ containing the positive real axis. $\Delta(\alpha)$ contains therefore also the points with $x>0, y=(2k+1)\pi$, and all the points with $y\neq(2k+1)\pi$, $x>-\alpha\sin^2\left(\frac{y}{2}\right)$.

In the s-plane, $(s=\sigma+it)$, we define the region $D(\lambda)$ $\left(0<\lambda<\frac{1}{2}\right)$ as the region containing the real positive axis of which the frontier is the curve $L(\lambda)$ given by

$$\sigma=\lambda \sin^2\left(\frac{t-\left[\frac{t}{2\pi}\right]2\pi}{4}\right), \tag{20}$$

where $[a]$ denotes the integer nearest to a (if $\frac{t}{2\pi}=n+\frac{1}{2}$, where n is an integer, $\left[\frac{t}{2\pi}\right]$ can be taken equal as well to n as to $n+1$, since the value of σ in (20) remains then the same). Each point of $L(\lambda)$ is clearly at a distance not greater than $\frac{\lambda}{2}$ from the t-axis; only the points $\left(\frac{\lambda}{2}, (2n+1)\pi\right)$ of $L(\lambda)$ are at the distance $\frac{\lambda}{2}$ from this axis.

Let us set, for $z \in \Delta(\alpha)$:

$$s=\varphi(z)=z-\log(1+e^{-z})+\log 2, \tag{21}$$

with $\log(1+e^{-x})>0$, $(x>0)$, the function $\log(1+e^{-z})$ being defined for other points of $\Delta(\alpha)$ by continuity.

LEMMA II. *If* $0<\lambda<\frac{1}{2}$, *there exists a positive* α *such that if* $z \in \Delta(\alpha)$, *then* $s=\varphi(z) \in D(\lambda)$.

We prove first that the points satisfying (19) are mapped on $\overline{D(\lambda)}$. We have:

$$\begin{aligned} |1+e^{\alpha \sin^2\left(\frac{y}{2}\right)-iy}|^2 &= 1+2e^{\alpha \sin^2\left(\frac{y}{2}\right)} \cos y + e^{2\alpha \sin^2\left(\frac{y}{2}\right)} \\ &= 1+\sum_0^\infty \frac{\left(2\alpha \sin^2\left(\frac{y}{2}\right)\right)^n}{n!}+2\cos y \sum_0^\infty \frac{\left(\alpha \sin^2\left(\frac{y}{2}\right)\right)^n}{n!} \\ &= 2(1+\cos y)+\alpha \sin^2\left(\frac{y}{2}\right).A(\alpha,y), \end{aligned}$$

where $A(\alpha,y)$ is given by a series which converges uniformly when $0<\alpha\leqq\alpha_1$, $-\infty<y<\infty$, $\alpha_1>0$ being any fixed quantity; and thus, for $0<\alpha<\alpha_1$, $|A(\alpha,y)|<M<\infty$. Hence, to every $\epsilon>0$ there corresponds an $\alpha(\epsilon)$ such that for $0<\alpha<\alpha(\epsilon)$:

$$|1+e^{\alpha\sin^2\left(\frac{y}{2}\right)-iy}|^2 \leqq 4-4\sin^2\left(\frac{y}{2}\right)+M\alpha\sin^2\left(\frac{y}{2}\right)$$
$$\leqq 4-4(1-\epsilon)\sin^2\left(\frac{y}{2}\right).$$

Hence if $\lambda<\lambda'<\frac{1}{2}$, we have for α sufficiently small:

$$|1+e^{\alpha\sin^2\left(\frac{y}{2}\right)-iy}|^2 \leqq 4\left(1-2\lambda'\sin^2\left(\frac{y}{2}\right)\right),$$

and

$$\log|1+e^{\alpha\sin^2\left(\frac{y}{2}\right)-iy}| \leqq \log 2+\frac{1}{2}\log\left(1-2\lambda'\sin^2\left(\frac{y}{2}\right)\right) \tag{22}$$
$$\leqq \log 2-\lambda'\sin^2\left(\frac{y}{2}\right).$$

If now z satisfies (19) we have for $s=\sigma+it=\varphi(z)$:

$$\sigma=\sigma(y)=-\alpha\sin^2\left(\frac{y}{2}\right)-\log|1+e^{\alpha\sin^2\left(\frac{y}{2}\right)-iy}|+\log 2,$$

and if α is sufficiently small, we have by (22):

$$\sigma=\sigma(y)\geqq-\alpha\sin^2\left(\frac{y}{2}\right)+\lambda'\sin^2\left(\frac{y}{2}\right)\geqq\lambda\sin^2\left(\frac{y}{2}\right). \tag{23}$$

On the other hand, we have for all y:

$$\langle\operatorname{Arg}(1+e^{\alpha\sin^2\left(\frac{y}{2}\right)-iy})\rangle\leqq\langle y\rangle, \tag{24}$$

where $\langle a\rangle=|a-\left[\frac{a}{2\pi}\right]2\pi|$. And, since, if $s=\varphi(z)=\varphi(x+iy)$,

$$t=y-\operatorname{Arg}(1+e^{-z})$$
$$\langle t\rangle\leqq\langle y\rangle+\langle\operatorname{Arg}(1+e^{-z})\rangle,$$

we see by (24) that $\langle t\rangle\leqq 2\langle y\rangle$, that is to say,

$$|t-\left[\frac{t}{2\pi}\right]2\pi|\leqq 2|y-\left[\frac{y}{2\pi}\right]2\pi|,$$

and

$$\sin^2\left(\frac{y}{2}\right)=\sin^2\left(\frac{\langle y\rangle}{2}\right)\geqq\sin^2\left(\frac{\langle t\rangle}{4}\right)=\sin^2\left(\frac{t-\left[\frac{t}{2\pi}\right]2\pi}{4}\right).$$

It follows now from (23) that if z satisfied (19) and if $s=\varphi(z)$ then:

$$\sigma \geqq \lambda \sin^2\left(\frac{t-\left[\frac{t}{2\pi}\right]2\pi}{4}\right), \tag{25}$$

that is to say, if z satisfies (19) with $\alpha>0$ sufficiently small: $\alpha \leqq \alpha_0$, then $s=\varphi(z) \in \overline{D(\lambda)}$.

Let us remark that $|1+e^{\alpha \sin^2\left(\frac{y}{2}\right)-iy}|$ (with y fixed), decreases as α decreases to $-\infty$. Therefore (22) and (23) hold also if z satisfies (19) with $-\infty<\alpha \leqq \alpha_0$. Since (24) is true for each fixed value of α, we see that (25) with the sign $>$ is true for $s=\varphi(z)$ where z is any point of $\Delta(\alpha_0)$ (open). We have thus proved that if $z \in \Delta(\alpha_0)$, then $s \in D(\lambda)$, which is the statement which was to be proved.

We now prove the following theorem [**8**]:

Theorem XI. *If $f(s)$ is holomorphic and bounded in a region $D(\lambda)$ with $0<\lambda<\frac{1}{2}$, if for σ sufficiently large*

$$f(s)=\sum_1^\infty a_n e^{-\lambda_n s},$$

where the sequence $\{\lambda_n\}$ is such that for no couple of two integers n, $m (n \neq m)$ is $\lambda_n-\lambda_m$ an integer and if $\sigma_A{}^f<\infty$, then the sequence $\{a_n\}$ is bounded.

If we write $s=\varphi(z)=z-\log(1+e^{-z})+\log 2$, we get from the relationship:

$$(1+e^{-z})^{\lambda_n}=\sum_0^\infty T_n{}^{(m)} e^{-mz},$$

where $\mathcal{R}(z)=x>0$, and where:

$$T_n^{(m)}=\frac{\lambda_n(\lambda_n-1)\cdots(\lambda_n-m+1)}{m!},$$

the equality:

$$\begin{aligned} \sum_n a_n e^{-\lambda_n s} &= \sum_n \frac{a_n}{2^{\lambda_n}} e^{-\lambda_n z}(1+e^{-z})^{\lambda_n} \\ &= \sum_n \frac{a_n}{2^{\lambda_n}} e^{-\lambda_n z} \sum_m T_n^{(m)} e^{-mz} = \sum_{r=1}^\infty b_r e^{-\mu_r z}. \end{aligned} \tag{26}$$

Here $\{\mu_n\}$ is a sequence of positive numbers tending to infinity and is such that to each r there corresponds a positive integer n and a non-negative integer m such that $\mu_r=\lambda_n+m$. Since we suppose that no two λ_n differ by an integer, to each r there corresponds only one such couple (n,m). We shall show that the last series in (26) converges absolutely for x sufficiently large, which means that the double series in this equality converges absolutely, since

$$b_r e^{-\mu_r z}=\frac{a_n}{2^{\lambda_n}}T_n^{(m)}e^{-(\lambda_n+m)z}, \tag{27}$$

where n and m are the two unique integers which correspond to r.

Since

$$\left|\frac{T_n^{(m)}}{T_n^{(m-1)}}\right|=\left|\frac{\lambda_r-m+1}{m}\right|\geqq 1,$$

if, and only if, $m\leqq\frac{\lambda_n+1}{2}$, we see that:

$$\max_m|T_n^{(m)}|=T_n^{\left(E\left(\frac{\lambda_n+1}{2}\right)\right)}, \tag{28}$$

where $E(c)$ denotes the largest integer not larger than c.

It follows from (28) that, on setting

$$\alpha_n=E(\lambda_n),\ \beta_n=E\left(\frac{\lambda_n+1}{2}\right):$$

$$C_{\alpha_n}^{\beta_n}\leqq\max_m|T_n^{(m)}|\leqq C_{\alpha_n+1}^{\beta_n} \tag{29}$$

(the left-hand inequality having a meaning when $\lambda_n\geqq 1$). But if α, β, with $\alpha>\beta$, are two integers tending to infinity, then

$$C_\alpha^\beta=\frac{\alpha!}{\beta!(\alpha-\beta)!}\backsim\frac{\sqrt{\alpha}}{\sqrt{2\pi}\sqrt{\beta(\alpha-\beta)}}\cdot\frac{\alpha^\alpha}{\beta^\beta(\alpha-\beta)^{\alpha-\beta}},$$

and if $\alpha=2\beta+O(1)$, $\beta\to\infty$, then

$$\log\frac{\alpha^\alpha}{\beta^\beta(\alpha-\beta)^{\alpha-\beta}}-\alpha\log 2=O(1).$$

Thus there exist two positive constants M_1, M_2 such that

$$\frac{M_1 2^{\lambda_n}}{\sqrt{\lambda_n}} < \max_m |T_n^{(m)}| < \frac{M_2 2^{\lambda_n}}{\sqrt{\lambda_n}}. \tag{30}$$

It follows then from (27) and (28) that, if $r=r(n)$ is such that $\mu_r = \lambda_n + E\left(\frac{\lambda_n+1}{2}\right)$, then

$$|a_n| = \frac{2^{\lambda_n}}{\max_m |T_n^{(m)}|} |b_{r(n)}| \leqq \frac{1}{M_1}\sqrt{\lambda_n}\,|b_{r(n)}|. \tag{31}$$

Moreover by (26):

$$\sum_{r=1}^{\infty} |b_r| e^{-\mu_r x} \leqq \sum_n \frac{|a_n|}{2^{\lambda_n}} e^{-\lambda_n x} \sum_m T_n^{(m)} e^{-mx}$$

$$\leqq M_2 \sum_n \frac{|a_n|}{\sqrt{\lambda_n}} e^{-\lambda_n x} \sum_m e^{-mx} = M_2 \sum \frac{|a_n|}{\sqrt{\lambda_n}} \frac{e^{-\lambda_n x}}{1-e^{-x}}.$$

The last series converges for $x > \max(\sigma_A{}^f, 0)$, hence $\sum b_r e^{-\mu_r z}$ has an abscissa of absolute convergence $(<\infty)$. By Theorem IX we may therefore write:

$$b_r = \lim_{T=\infty} \frac{1}{T}\int_0^T F(x_1+iy) e^{\mu_r(x_1+iy)}\, dy, \tag{32}$$

where $F(z) = \sum b_r e^{-\mu_r z}$, and where x_1 is sufficiently large.

By Lemma II, since $f(s)$ is holomorphic and bounded in some region $D(\lambda)$ with $0<\lambda<\frac{1}{2}$, $F(z)$ is holomorphic and bounded in $\Delta(\alpha)$ where $\alpha>0$ is chosen sufficiently small.

On denoting, for r sufficiently large, by $C_r(\alpha)$ the curve in the z-plane composed of segments

$$\begin{gathered} y=(2k+1)\pi \\ -\alpha+\frac{1}{\mu_r} \leqq x \leqq 0, \end{gathered} \tag{33}$$

and the curve

$$x = -\alpha \sin^2\left(\frac{y}{2}\right) + \frac{1}{\mu_r}, \tag{34}$$

we see, by an immediate application of Cauchy's theorem, that from (32) follows the equality

$$(35) \qquad b_r = \lim_{T=\infty} \frac{1}{Ti} \int_{C_r(\alpha,T)} F(z) e^{\mu_r z} dz,$$

where $C_r(\alpha, T)$ is the part of the curve $C_r(\alpha)$ which begins at the point $\left(\frac{1}{\mu_r}, 0\right)$ and ends at the point with ordinate T, the segments of form (33) with $0 < (2k+1)\pi < T$, which therefore belong to $C_r(\alpha, T)$, being counted twice, first running from $\left(-\alpha+\frac{1}{\mu_r}, (2k+1)\pi\right)$ to $(0, (2k+1)\pi)$ and afterwards in the opposite sense $\left(\text{the direction on } C_r(\alpha, T) \text{ is defined by the fact that it begins at } \left(\frac{1}{\mu_r}, 0\right)\right)$. Formula (35) follows from the fact that the integrals of $F(z)e^{\mu_r z}$ over the two segments $\left(\frac{1}{\mu_r} \leqq x \leqq x_1, t=0\right)$ and the segment with $t = T$ situated between $C_r(\alpha)$ and the line $x = x_1$ are bounded.[1]

It follows then from (35) that

$$|b_r| \leqq M \left[\int_0^\pi e^{\left(-\alpha \sin^2\left(\frac{y}{2}\right)+\frac{1}{\mu_r}\right)\mu_r} dy + \int_{-\alpha+\frac{1}{\mu_r}}^0 e^{\mu_r x} dx \right],$$

where M is a positive constant. We have thus:

$$(36) \qquad |b_r| \leqq M_1 \int_0^\pi e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} dy + \frac{M}{\mu_r} (1 - e^{1-\alpha\mu_r}).$$

For every fixed r sufficiently large, we choose η such that $\frac{1}{\mu_r} < \alpha^2 \sin^2 \frac{\eta}{2} < \frac{2}{\mu_r}$, and write

[1]On the two sides of the segments of form (33) $F(z)$ is defined by continuity from its values in $\Delta(\alpha)$. It is obvious that, for $z = (2k+1)\pi i$, $F(z)$ should be taken equal to $\lim_{\sigma=\infty} f(s) = a_1 \lim_{\sigma=\infty} e^{-\lambda_1 s} (=0 \text{ if } \lambda_1 > 0, = a_1 \text{ if } \lambda_1 = 0)$. This definition by continuity is possible since if z in $\Delta(\alpha)$ tends to a point of a segment (33) then $s = \varphi(z)$ tends either to a point of $\Delta(\lambda)$ or to $+\infty$.

$$\int_0^{\pi} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} dy < 2\left(\int_0^{\eta} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} dy + \int_{\eta}^{\frac{\pi}{2}} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} dy\right).$$

The first of the integrals on the right hand is smaller than η. For the second integral we have the immediate estimate:

$$\int_{\eta}^{\frac{\pi}{2}} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} dy < \frac{\sqrt{2}}{\sin\frac{\eta}{2}} \int_0^{\frac{\pi}{2}} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} d\left(\sin^2\left(\frac{y}{2}\right)\right)$$

$$= \frac{\sqrt{2}}{\sin\frac{\eta}{2}} \int_0^{\frac{1}{2}} e^{-\alpha\mu_r u} du = O\left(\frac{1}{\sqrt{\mu_r}}\right).$$

It follows thus from (36) that

$$|b_r| = O\left(\frac{1}{\sqrt{\mu_r}}\right),$$

and if $r = r(n)$, that is to say, if $\mu_r = \lambda_n + E\left(\frac{\lambda_n + 1}{2}\right)$, then

$$|b_{r(n)}| = O\left(\frac{1}{\sqrt{\lambda_n}}\right).$$

From (31) it follows then that $|a_n|$ is bounded $(n \geqq 1)$ and our theorem is proved.

We shall now recall a definition concerning sequences. The elements of a sequence $\{p_n\}$ $(n \geqq 1)$ are said to be *linearly independent* if from each equality of the form

$$\sum_1^n A_m p_m = 0,$$

where the A_m are integers, it follows that all these integers are equal to zero.

We are now in a position to prove the following theorem:

THEOREM XII. *If $f(s)$ is holomorphic in a region $D(\lambda)$, with $0 < \lambda < \frac{1}{2}$, and if, for σ sufficiently large,*

$$f(s) = \sum_1^{\infty} a_n e^{-\lambda_n s}$$

where $\sigma_A{}^f < \infty$, *where the sequence* $\{\lambda_n\}$ *is such that the sequence composed of all the* λ_n *and unity is composed of linearly independent elements, and if there exist a constant* $a \neq 0$ *and a positive quantity* ϵ *such that in* $D(\lambda)$:

(37) $$|f(s) - a| > \epsilon,$$

then $|a| > \epsilon$, *and*

(38) $$|a_n| \leqq |a| - \epsilon, \quad (n \geqq 1).$$

Consider the function $F(s) = \dfrac{1}{a - f(s)}$. If $\sigma > \sigma_A{}^f$ we have in $D(\lambda)$:

$$|f(s)| < \sum |a_n| e^{-\lambda_n \sigma} = e^{-\lambda_1 \sigma} \sum |a_n| e^{-(\lambda_n - \lambda_1)\sigma},$$

and since, obviously, $\lambda_1 \neq 0$, we see that $\lim\limits_{\sigma = \infty} f(s) = 0$ uniformly with respect to t. This proves that $|a| > \epsilon$, and that the function $F(s)$, which is obviously holomorphic and bounded in $D(\lambda)$, is also given for σ sufficiently large (such that $\sum |a_n| e^{-\lambda_n \sigma} < |a|$) by the equality:

$$F(s) = \frac{1}{a - \sum\limits_n a_n e^{-\lambda_n s}} = \frac{1}{a} \sum_m \left(\frac{\sum\limits_n a_n e^{-\lambda_n s}}{a} \right)^m = \sum_k c_k e^{-L_k s},$$

where the sequence $\{L_k\}$ of non-negative, increasing quantities tends to infinity. Since each L_k is of the form

$$L_k = A_1 \lambda_1 + A_2 \lambda_2 + \cdots + A_m \lambda_m,$$

where the A_i are integers, we see, by the hypothesis on the sequence $\{\lambda_n\}$, that if $k_1 \neq k_2$, $L_{k_1} - L_{k_2}$ is not an integer. By Theorem XI we see then that there exists a constant $M < \infty$, such that $|C_k| < M (k \geqq 0)$. On the other hand, it is readily seen that the sequence $\{L_k\}$ contains all the quantities of the form $m\lambda_n$, where m and n are arbitrary positive integers, and that if $L_k = m\lambda_n$ then $C_k = \dfrac{a_n^m}{a^{m+1}}$. Therefore for n arbitrary, fixed, and for every positive m we have:

$$\frac{|a_n|^m}{|a|^{m+1}} < M,$$

and

$$|a_n| < |a|^{\frac{m+1}{m}} M^{\frac{1}{m}} \ (m \geqq 1).$$

Hence

$$|a_n| \leqq \lim_{m=\infty} \left(|a|^{\frac{m+1}{m}} M^{\frac{1}{m}}\right) = |a|.$$

In this inequality $|a|$ can be replaced by $|a| - \epsilon$, since from $|f(s) - a| > \epsilon$ in $D(\lambda)$ it follows immediately that, if $a = |a|e^{i\varphi}$, then, for each $0 < \theta < 1$,

$$|f(s) - (|a| - \theta\epsilon)e^{i\varphi}| > (1 - \theta)\epsilon$$

in $D(\lambda)$, and therefore, in what precedes $|a|$ can be replaced by $|a| - \theta\epsilon$. That is to say, we have $|a_n| \leqq |a| - \theta\epsilon$, and since θ has only to satisfy the inequality $0 < \theta < 1$, our theorem is proved.

Theorem XII may also be stated as follows:

THEOREM XIII. *If the sequence* $\{\lambda_n\}$ *satisfies the same conditions as in Theorem XII, if* $f(s)$ *is holomorphic in* $D(\lambda)$, $\left(0 < \lambda < \frac{1}{2}\right)$ *and is given for* σ *sufficiently large by* $f(s) = \sum a_n e^{-\lambda_n s}$ *with* $\sigma_A{}^f < \infty$, *and if*

$$A = \underset{n \geqq 1}{\text{l. u. b.}} |a_n|,$$

then the values which $f(s)$ *takes when* $s \in D(\lambda)$ *are everywhere dense in the circle with radius* A *around the origin.*

Theorems XI and XII were proved by the author [**9**]. They bear a character analogous to that of the well known theorem of Weierstrass by which an entire function takes values everywhere dense in the whole plane. A sharper form of Theorem XIII was later given by Aronszajn [**1**,**2**]. This author, on using Theorem XII and the modular function, obtained a theorem which is in character analogous to Picard's theorem on entire (or meromorphic) functions, by which each such function takes all the possible values except at most one finite value (two for meromorphic).

Aronszajn's theorem can be stated in the following manner:

THEOREM XIV. *If the sequence* $\{\lambda_n\}$ *satisfies the same conditions as in Theorem XII, if* $f(s)$ *is holomorphic in* $D(\lambda)$ *with* $0<\lambda<\frac{1}{2}$ *and is given for* σ *sufficiently large by* $\sum a_n e^{-\lambda_n s}$ *with* $\sigma_A{}^f<\infty$, *then* $f(s)$ *takes in* $D(\lambda)$ *all the possible values situated in the circle with radius* $A=\sum_1^\infty |a_n|$ *except at most the value zero and one other value.*

If $a\neq b$ then it is known [**15**] that there exists, in the complex plane a function $\chi(u)$ with the following properties: α) $\chi(u)$ is regular at the origin and is therefore given, in its neighborhood, by $\chi(u)=\sum A_n u^n$, β) $\chi(u)$ can be continued analytically on each curve which does not pass through a or b, the points $u=a$, $u=b$ being branch points for $\chi(u)$, γ) $|\chi(u)|<1$, δ) obviously

$$\limsup_{n=\infty} |A_n|^{\frac{1}{n}}=\frac{1}{\min(|a|,|b|)}=\frac{1}{c}. \tag{39}$$

Suppose now that, contrary to the assertion, $f(s)$ does not take two values a, b, such that $a\neq b$, $a\neq 0$, $b\neq 0$, $|a|<A$, $|b|<A$, and let us consider the function $F(s)=\chi[f(s)]$, where $\chi(u)$ is the function defined above. $F(s)$ is holomorphic in $D(\lambda)$. If σ' is chosen such as to have $\sum|a_n|e^{-\lambda_n\sigma'}<c$, we see that $\sum_m |A_m|(\sum|a_n|e^{-\lambda_n\sigma'})^m$ converges, and therefore $F(s)$ is given for σ sufficiently large by a Dirichlet series:

$$F(s)=\sum A_m(f(s))^m$$

$$=\sum_n A_n \sum_{\substack{m_1+\cdots+m_p=n\\ 1\leq l_i<\infty\\ i=1,2,\cdots p}} \frac{n!}{m_1!\cdots m_p!}(a_{l_1})^{m_1}\cdots(a_{l_p})^{m_p}e^{-s(m_1\lambda_{l_1}+\cdots+m_p\lambda_{l_p})}$$

$$=\sum c_k e^{-L_k s},$$

which admits an abscissa of absolute convergence. We have in $D(\lambda)$: $|F(s)|<1$. Here, as in the proof of Theorem XII, we have $L_{k_1}-L_{k_2}\neq$integer, and, by Theorem XI, we should have $|c_k|<M<\infty\,(k\geqq 1)$. But, if $L_k=m_1\lambda_{l_1}+\cdots+m_p\lambda_{l_p}$, then $c_k = \frac{n!}{m_1!\cdots m_p!}(a_{l_1})^{m_1}\cdots(a_{l_p})^{m_p}$, and thus:

(40) $$\frac{n!}{m_1!\cdots m_p!}|a_{l_1}|^{m_1}\cdots|a_{l_p}|^{m_p} < \frac{M}{|A_n|}.$$

Let p and $l_1, \cdots l_p$ be fixed, and let us take p positive constants: $\alpha_1, \alpha_2, \cdots \alpha_p$ with $\sum_1^p \alpha_i = 1$, and, for each positive integer n, let us set $m_j = E(n\alpha_j)(1 \leqq j \leqq p-1)$, $m_p = n - \sum_1^{p-1} m_j$. Then, by Stirling's formula,

$$\lim_{n=\infty} \sqrt[n]{\frac{n!}{m_1!\cdots m_p!}} = \alpha^{-\alpha_1}\cdots\alpha_p{}^{-\alpha_p},$$

and by (40):

(41)

$$\alpha_1{}^{-\alpha_1}\cdots\alpha_p{}^{-\alpha_p}|a_{l_1}|^{\alpha_1}\cdots|a_{l_p}|^{\alpha_p} \leqq \frac{1}{\limsup|A_n|^{\frac{1}{n}}} = c = \min(|a|,|b|).$$

This is seen on putting in (40) $n = n_i$ where $\{n_i\}$ is such that $\lim_{i=\infty}|A_{n_i}|^{\frac{1}{n_i}} = \limsup|A_n|^{\frac{1}{n}}$, and on making i tend to infinity. Since we can suppose that the a_n are all distinct from zero we shall put:

$$\alpha_j = \frac{|a_{l_j}|}{|a_{l_1}| + \cdots + |a_{l_p}|}(1 \leqq j \leqq p).$$

Then $\alpha_j^{\alpha_j} = \left(\frac{|a_{l_j}|}{|a_{l_1}| + \cdots + |a_{l_p}|}\right)^{\alpha_j}$, and (41) gives immediately

$$|a_{l_1}| + \cdots + |a_{l_p}| \leqq \min(|a|, |b|),$$

and since the l_j are arbitrary we shall have

$$\sum_1^\infty |a_n| \leqq \min(|a|, |b|) < A,$$

which gives a contradiction.

Aronszajn proves even more general theorems, considering meromorphic functions $f(s)$, and the elementary methods which we just used are susceptible of furnishing results [**1**] which contain as particular cases many theorems of H. Bohr [**1**] concerning almost periodic functions of a complex variable. It is not, however, our intention to explore here any farther this interesting and important field.

Obviously Theorem XIV is more precise than Theorem XIII, but we thought it useful to prove both, since, although Theorem XIV is, likewise Theorem XIII, based on Theorem XI, the passage from Theorem XI to Theorem XII or XIII is still elementary, whereas the use of the modular function is necessary for the passage from Theorem XI to Theorem XIV.

It is of course impossible to have the conclusions of either Theorems XI, XII (or XIII) and XIV without specific hypotheses on the sequences $\{\lambda_n\}$. For instance, the principal branch of $f_1(s) = \sum_1^\infty (-1)^n n e^{-ns}$ is bounded in $D(\lambda)$ with an arbitrary $\lambda\left(0<\lambda<\frac{1}{2}\right)$, $\sigma_A{}^{f_1}=0$, and yet the $|a_n|$ are not bounded. The principal branch of $f_2(s)=\sum_1^\infty (-1)^n e^{-ns}$ is also bounded in $D(\lambda)$, $\sigma_A{}^{f_2}=0$, l. u. b. $|a_n|=1$ and yet the values taken by $f_2(s)$ not only in $D(\lambda)$ but even in the half-plane $\sigma>0$ do not penetrate the part of the circle $|u|<1$ with $\mathcal{R}(u)\geqq\frac{1}{2}$.